Stability Analysis

Before designing a controller we first need to determine the stability of the control system.

1. <u>Dynamic System</u>: Consider a general nonlinear system

$$\dot{x} = f(x), \qquad x \in R^n \qquad (1)$$

where

$$f : D \to \mathbb{R}^n$$

Domain D can be global or locally defined as

$$D = \mathbb{R}^n, \text{ global}$$

or

$$D = B(h) = \{ x \in \mathbb{R}^n : |x| < h \}, \text{ local}$$

where $|\cdot|$: norm in $\mathbb{R}^n$, e.g.,

$$|x| = \sqrt{x^T x}$$

<u>Initial Value Problem</u>:

$$\dot{x} = f(x(t)), \qquad x(0) = x_0 \qquad (2)$$

<u>Assume</u>: A unique solution exists.

<u>Note</u>: $\phi(t, x_0)$ is a solution of (1)

if $\dot{\phi}(t, x_0) = f(\phi(t, x_0)), \quad \phi(0, x_0) = x_0$

<u>Defn</u>: A point $x_e \in \mathbb{R}^n$ is called an <u>equilibrium point</u> of (2) if $f(x_e) = 0 \quad \forall t \geq 0$.

<u>Defn</u>: An equilibrium point x_e is an "isolated equilibrium point", if $\exists$ an $h' > 0 \ni$

$$B(x_e, h') = \{ x \in \mathbb{R}^n : |x - x_e| < h' \}$$

contains no other equilibrium points.

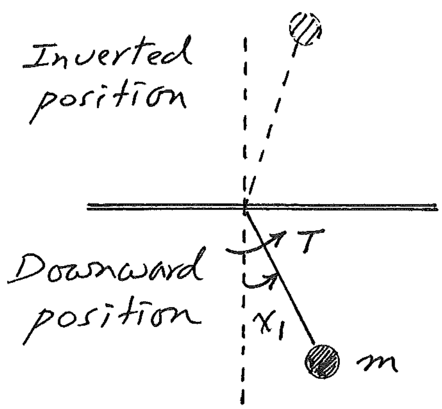
<u>Assume</u>: The equilibrium of interests is an isolated equilibrium located at the origin.

<u>Note</u>: If $x_e \neq 0$, then let $\bar{x} = x - x_e$,

$\Rightarrow$ $\bar{x} = 0$ is an equilibrium of

$$\dot{\bar{x}} = \bar{f}(\bar{x}) = f(\bar{x} + x_e).$$

<u>Example</u>:



<u>Pendulum</u>:

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -\frac{g}{l}\sin(x_1) - \frac{k}{m}x_2 + \frac{1}{ml^2}T$$

For $T=0$, equilibrium pts:

$$[0, 0]^T, \quad [\pi, 0]^T$$

<u>Inverted position</u>: Translate the equilibrium:

$$\bar{x} = x - [\pi, 0]^T$$

Then

$$\dot{\bar{x}}_1 = \dot{x}_1 = \bar{x}_2 = f_1(\bar{x})$$

$$\dot{\bar{x}}_2 = \dot{x}_2 = -\frac{g}{l}\sin(\bar{x}_1 + \pi) - \frac{k}{m}\bar{x}_2 + \frac{1}{ml^2}T$$

$$= \frac{g}{l}\sin\bar{x}_1 - \frac{k}{m}\bar{x}_2 + \frac{1}{ml^2}T$$

We can drop the bar "$-$" for simplicity in notation.

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Stability Definitions:

Def. The equilibrium $x_e = 0$ of (1) is stable (in the sense of Lyapunov) if for every $\varepsilon > 0$ $\exists$ a $\delta(\varepsilon) > 0$ $\ni$ $|\bar{\phi}(t, x_0)| < \varepsilon$ $\forall t > 0$ whenever $|x_0| < \delta(\varepsilon)$.

Def. The equilibrium $x_e = 0$ of (1) is "asymptotically" stable if it is stable and $\exists$ $\eta > 0$ $\ni$

$$\lim_{t \to \infty} \bar{\phi}(t, x_0) = 0 \quad \text{whenever} \quad |x_0| < \eta.$$

Def. The set $X_d \subset \mathcal{R}^n$ of all $x_0 \in \mathcal{R}^n$ $\ni$ $\bar{\phi}(t, x_0) \to 0$ as $t \to \infty$ is called the "domain of attraction" of the equilibrium point $x_e = 0$ of (1).

Def. The equilibrium $x_e = 0$ is "globally asymptotically stable" if $X_d = \mathcal{R}^n$.

Example:

$$\dot{x} = -2x$$

$D = \mathcal{R}^1$, real line.

Equilibrium point: $0 = -2x_e \rightarrow x_e = 0$

Solution: $\bar{\phi}(t, x_0) = x_0 e^{-2t} \rightarrow 0$ as $t \to \infty$

: stable equilibrium point

: globally asymptotically stable.

Unfortunately, in a complex system the solution is not easily obtained, and we need a method of determining stability without knowing the solution of the dynamic system model. This leads to the Lyapunov's direct method.

Lyapunov's Direct Method.

We depend on the existence of an appropriate Lyapunov Function:

$$V : D \to \mathcal{R}, \text{ real}$$

where $D = \mathcal{R}^n$ (global) or $D = B(h)$ (local)

Assuming (1) is differentiable with respect to all its arguments, the derivative of V with respect to t along the trajectories of (1) is

$$\dot{V}(x(t)) = \nabla(V(x(t))^T f(x(t))$$

where

$$\nabla V(x(t)) = \left[\frac{\partial V}{\partial x_1}, \frac{\partial V}{\partial x_2}, \cdots, \frac{\partial V}{\partial x_n}\right]^T$$

: gradient (vector) of V with respect to

Theorem: Let $x_e = 0$ be an equilibrium for (1). Let $V: B(h) \to \mathcal{R}$ be continuously differentiable function on $B(h) \ni V(0) = 0$ and $V(x) > 0$ in $B(h) - \{0\}$, and $\dot{V}(x) \leq 0$ in $B(h)$.
Then $x_e = 0$ is stable. If, in addition, $\dot{V}(x) < 0$ in $B(h) - \{0\}$, then $x_e = 0$ is asymptotically stable.

Theorem: Let $x_e = 0$ be an equilibrium for (1). Let $V: \mathcal{R}^n \to R$ be a continuously differentiable function $\ni V(0) = 0$ and $V(x) > 0$ for all $x \neq 0$, $|x| \to \infty$ implies that $V(x) \to \infty$, and $\dot{V}(x) < 0 \ \forall \ x \neq 0$.
Then $x_e = 0$ is globally asymptotically stable.

<u>Example:</u> $\dot{x} = -2x^3$

$x_e = 0$ is an equilibrium.

Choose

$$V(x) = \frac{1}{2}x^2$$

Note that this choice of Lyapunov function satisfies all the hypotheses.

$$\dot{V} = \frac{\partial V}{\partial x}\frac{dx}{dt} = x\dot{x} = -2x^4 < 0 \quad \forall x \neq 0.$$

Therefore the system is asymptotically globally stable.

<u>Discrete-Time Systems:</u> Consider the nonlinear discrete-time system

$$x(k+1) = f(x(k)), \qquad x \in \mathbb{R}^n \qquad (3)$$

where $f: D \to \mathbb{R}^n$

with $D = \mathbb{R}^n$ or $D = B(h)$ for some $h > 0$.

<u>Theorem:</u> The equilibrium $x_e = 0$ of (3) is globally asymptotically stable if $\exists$ a function $V(x) \ni$ the following hold $\forall x \in \mathbb{R}^n$:

1. $V(x) \geq 0$ except at $x = 0$ where $V(x) = 0$,
2. $V(x) \to \infty$ if $|x| \to \infty$, and
3. $V(x(k+1)) - V(x(k)) < 0$

If these condition only hold locally, then we only obtain <u>asymptotic stability</u>, and the corresponding domain D is called the region of asymptotic stability.

Example:

$$x(k+1) = a\,x(k)$$

$x_e = 0$: an isolated equilibrium.

Choose $V = x^2$. This satisfies conditions 1 & 2.

For condition 3,

$$V(x(k+1)) - V(x(k)) = x^2(k+1) - x^2(k)$$

$$= a^2x^2 - x^2 = (a^2-1)x^2$$

If $(a^2-1) < 0$ or $a^2 < 1$ or $a \in (-1, 1)$, then $x_e = 0$ is a globally asymptotically stable equilibrium point.

Neural Control:

Consider a control system

$$\dot{x} = f(x) + g u \qquad (4)$$

where $x(t) \in R$, $g > 0$, unknown but fixed scalar.

f: smooth function and $f(0) = 0$ and

$$|f(x)| < \alpha |x|$$

Design a neural controller

$$u = F(x)$$

such that the equilibrium $x_e = 0$ is globally asymptotically stable.

We choose a Lyapunov function

$$V = \frac{1}{2}x^2$$

Then $\dot{V} = x\dot{x} = x f(x) + g x u = x f(x) + g x F(x)$,

and $\dot{V} \le |x||f(x)| + g\,x\,F(x)$

$$\le \alpha x^2 + g x F(x) \qquad (5)$$

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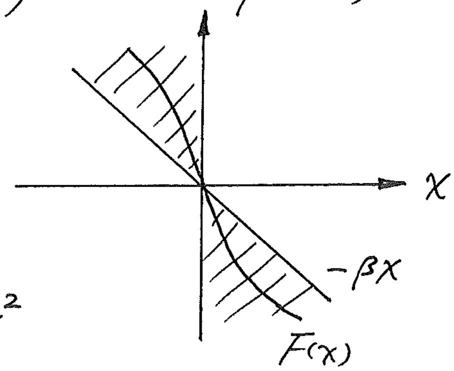
We want the 2nd term in (5) become negative so that $\dot{V} < 0$ for $x \neq 0$.

Design $F(x) \ni F(0) = 0$ and for some $\beta > 0$,

$$\left.\begin{aligned} F(x) &> -\beta x, \quad x < 0 \\ F(x) &< -\beta x, \quad x > 0 \end{aligned}\right\} \quad (6)$$



Then, for $x > 0$,

$$\dot{V} \le \alpha x^2 + g x(-\beta x) = (\alpha - g\beta)\, x^2$$

for $x < 0$, $\quad x F(x) < -\beta x^2$

$$\dot{V} \le \alpha x^2 + g(-\beta x^2) = (\alpha - g\beta)\, x^2$$

Hence, if $\alpha - g\beta < 0$ or $\beta > \frac{\alpha}{g}$, then

$x_e = 0$ is globally asymptotically stable.

<u>NOTE:</u>

$$\dot{x} = f(x) + g \cdot F(x)$$

If $x > 0$, $\quad F(x) < 0$: $\quad x$ decreases to $x_e = 0$

If $x \; 0$, $\quad F(x) > 0$: $\quad x$ increases to $x_e = 0$.

<u>Remark:</u> NN's are often saturated and (6) is not satisfied globally.

Thus, $x_e = 0$ is only asymptotically stable (local) or $\exists$ a region of asymptotic stability.